Essential Concepts

Thank you for coming to this experimental presentation.

The Executive Board of the CONCEPTS series asked me to deliver three short talks on “Essential Concepts”.

The idea is not to present new research results, but to illuminate some aspects of the theory which perhaps deserve more attention.

Today’s presentation is mildly technical. It demonstrates a method for creating a concept lattice from two known ones.

Essential Concepts:

Subdirect product constructions

Bernhard Ganter

Ernst-Schröder–Zentrum Darmstadt

Montpellier, September 2, 2026

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination.

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

You can remove a few crosses.

	×	×	×	×		×				
	×	×	×	×	×	×	×			
	×	×	×			×				
	×	×		×	×	×	×			
	×		×				×			
	×		×	×		×		×	×	
	×	×	×	×	×			×		
	×	×	×	×	×	×				×
	×		×	×		×	×		×	

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

You can remove a few crosses.

This can make the concept lattice smaller, but also larger. If you do it correctly and produce *closed subrelations*, you'll end up with exactly the complete sublattices.

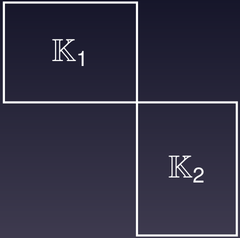
	×	~~×~~	×	~~×~~		×				
	×	×	~~×~~	~~×~~	~~×~~	×	×			
	~~×~~	×	×			×				
	×	~~×~~		×	~~×~~	~~×~~	×			
	×		~~×~~				×			
	×		×	×		×		×	×	
	×	~~×~~	×	~~×~~	×			~~×~~		
	×	×	~~×~~	~~×~~	×	×				~~×~~
	×		×	×		×	~~×~~		×	

Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

You can combine two crosstables.



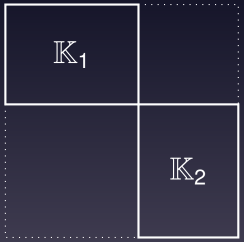
Playing around with formal contexts

One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

You can combine two crosstables.

In most cases you need to add something to create a new formal context.



Playing around with formal contexts

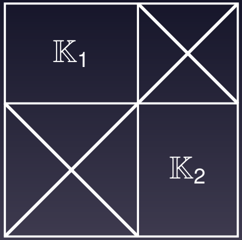
One of the great things about Formal Concept Analysis is that you can work with complex algebraic structures simply by tinkering with crosstables.

The secret to success is a good mix of playfulness and expert determination. Here are a few simple things to try:

You can combine two crosstables.

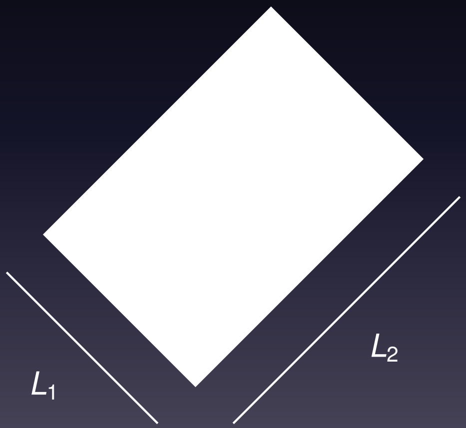
In most cases you need to add something to create a new formal context.

The **context sum** corresponds to the **direct product of the lattices.**



The direct product

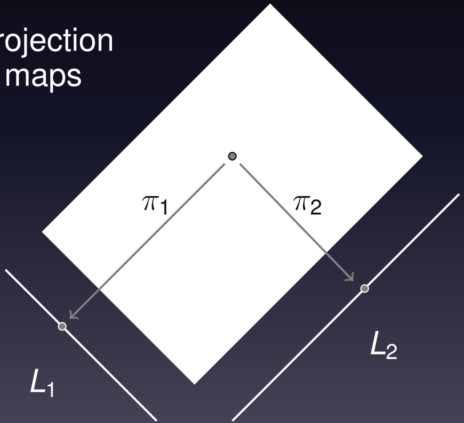
$$L_1 \times L_2 := \{(x, y) \mid x \in L_1, y \in L_2\}.$$



The direct product

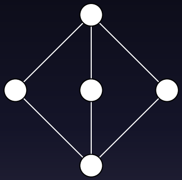
$$L_1 \times L_2 := \{(x, y) \mid x \in L_1, y \in L_2\}.$$

Projection maps

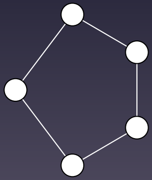


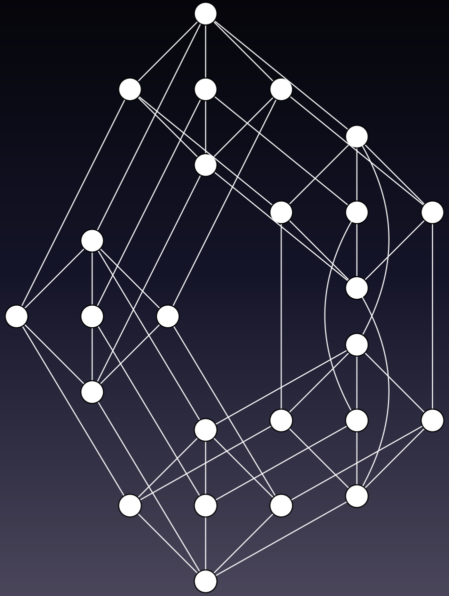
Example

The diagram on the right shows the direct product of



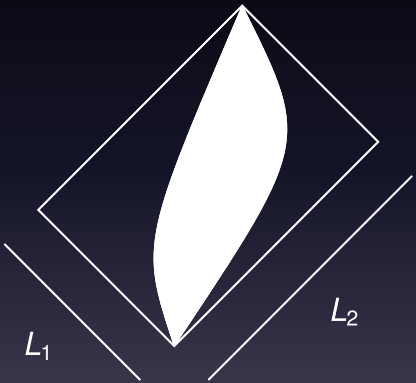
and





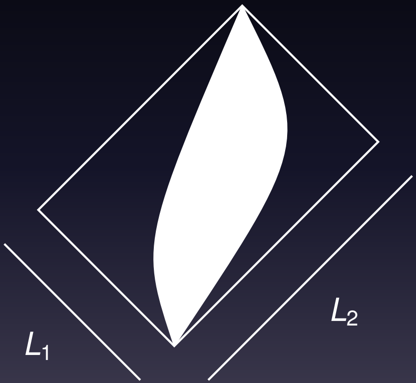
A subdirect product . . .

. . . is a sublattice of a direct product for which the projections are onto.



A subdirect product . . .

. . . is a sublattice of a direct product for which the projections are onto.

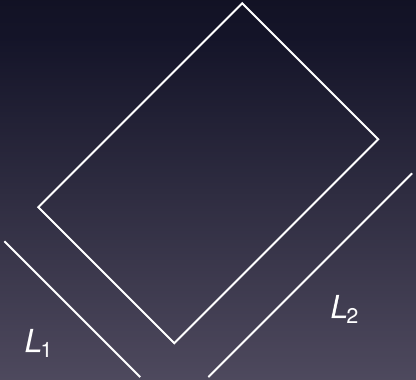


The direct product $L_1 \times L_2$ is unique, but there are usually many subdirect products.

Subdirect product construction

To turn the subdirect product into a *construction*, we first choose a (sufficiently large) set P, here symbolized as

$$P := \{\textcolor{blue}{\bullet}, \textcolor{green}{\bullet}, \textcolor{red}{\bullet}\}.$$

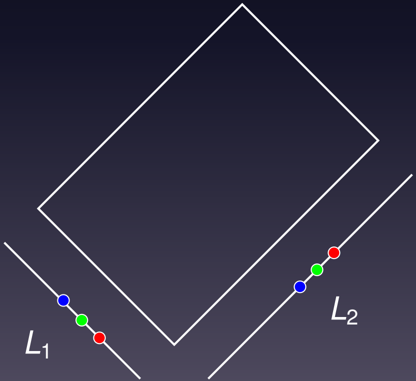


Subdirect product construction

To turn the subdirect product into a *construction*, we first choose a (sufficiently large) set P, here symbolized as

$$P := \{\bullet, \bullet, \bullet\}.$$

Then P is used to label a generating set in L_1 and a generating set in L_2, respectively.

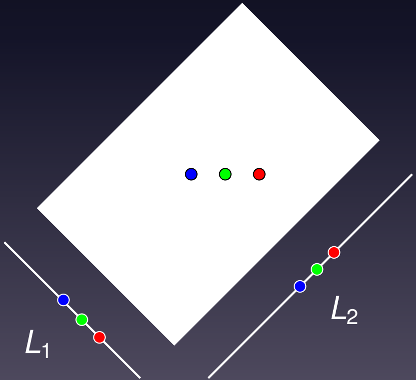


Subdirect product construction

To turn the subdirect product into a *construction*, we first choose a (sufficiently large) set P, here symbolized as

$$P := \{\bullet, \bullet, \bullet\}.$$

This naturally results in P-labeled elements of the direct product.

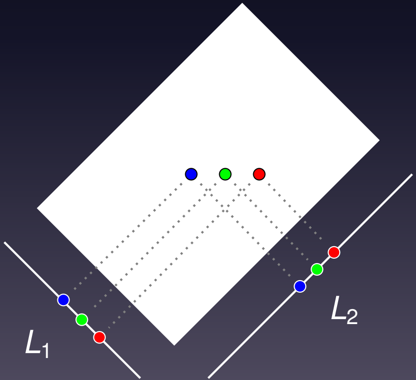


Subdirect product construction

To turn the subdirect product into a *construction*, we first choose a (sufficiently large) set P, here symbolized as

$$P := \{\bullet, \bullet, \bullet\}.$$

This naturally results in P-labeled elements of the direct product.

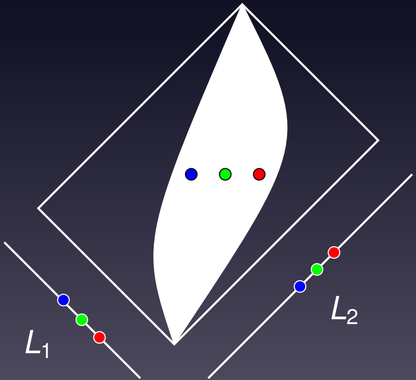


Subdirect product construction

To turn the subdirect product into a *construction*, we first choose a (sufficiently large) set P, here symbolized as

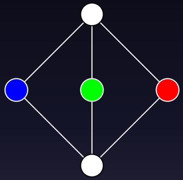
$$P := \{\color{blue}\bullet, \color{green}\bullet, \color{red}\bullet\}.$$

The sublattice of the direct product, generated by these elements, is the resulting P**-product**.

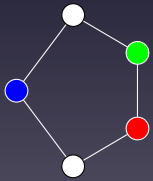


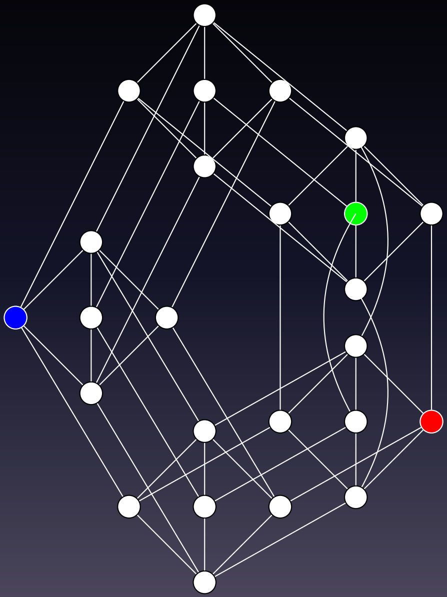
Example

The diagram on the right shows the direct product of



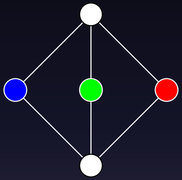
and



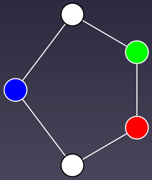


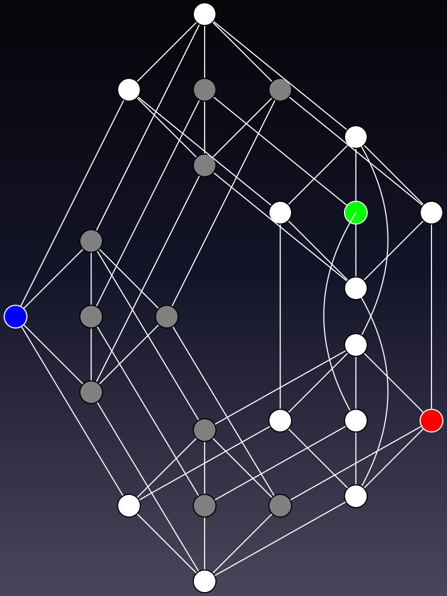
Example

The diagram on the right shows the *P*–product of

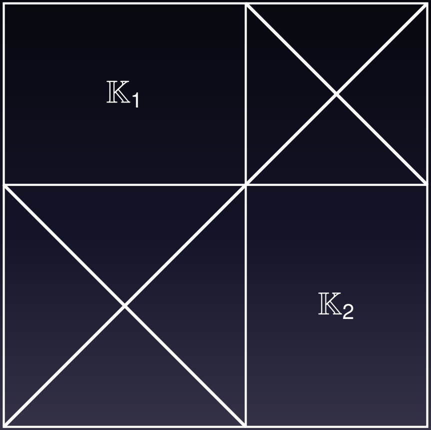


and





The context construction: P-Fusion



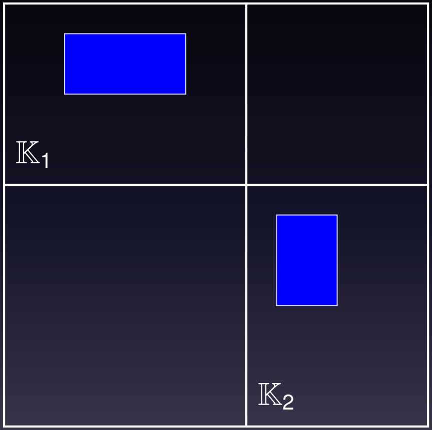
We are looking for a compatible subrelation of the context sum that encodes the P-product.

The context construction: *P*-Fusion

$\mathbb{K}_1$	$\emptyset$
$\emptyset$	$\mathbb{K}_2$

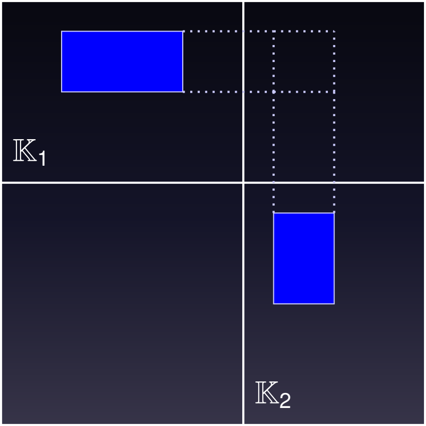
$\mathbb{K}_1$ and $\mathbb{K}_2$ remain unchanged. The other incidences are removed.

The context construction: *P*-Fusion



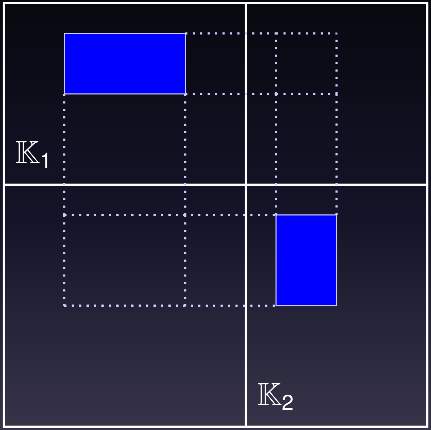
For each element $p \in P$, find the generators labeled p.

The context construction: *P*-Fusion



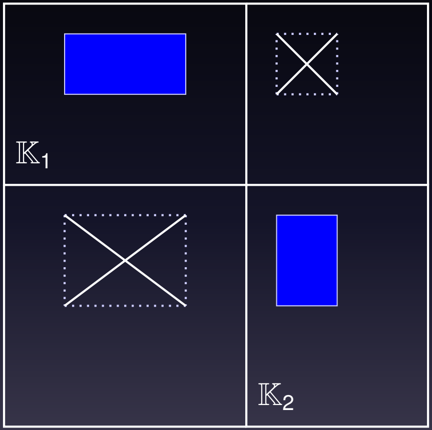
Determine the rectangles $\mathbb{K}_1$-extent $\times$ $\mathbb{K}_2$-intent . . .

The context construction: *P*-Fusion



. . . and $\mathbb{K}_2$-extent $\times$ $\mathbb{K}_1$-intent.

The context construction: *P*-Fusion



Repeat this for all $p \in P$ and then close the bonds.

Summary

Denote a P-lattice by (L, λ), where L is the lattice and $\lambda : P \to L$ labels a generating set.

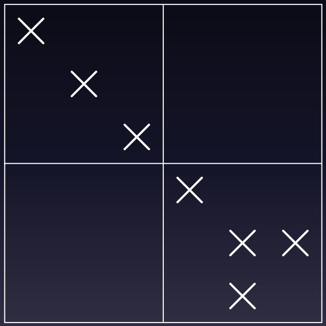
Given two P-lattices (L_1, λ_1) and (L_2, λ_2), there is a natural labeling $\lambda : P \to L_1 \times L_2$, defined by

$$\lambda(p) := (\lambda_1(p), \lambda_2(p)).$$

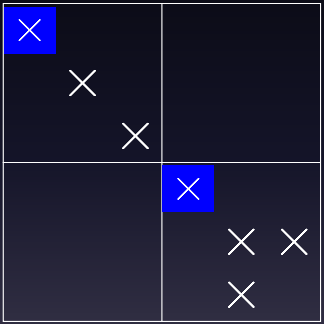
The (complete) sublattice generated by these labeled elements is both a P-lattice and a subdirect product of L_1 and L_2.

It is called the ***P*-product** of (L_1, λ_1) and (L_2, λ_2).
The context construction is the ***P*-fusion**.

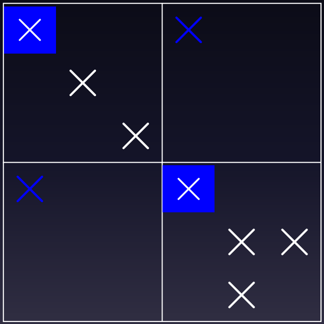
The P-fusion for the example



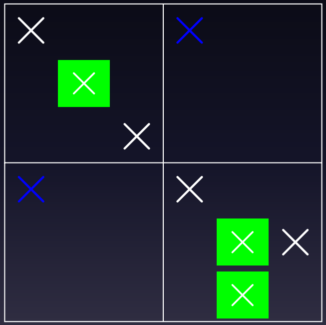
The P-fusion for the example



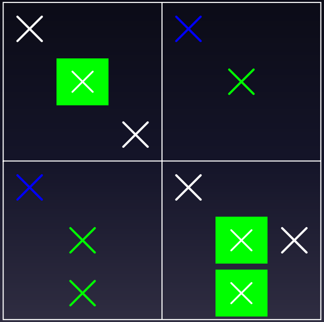
The P-fusion for the example



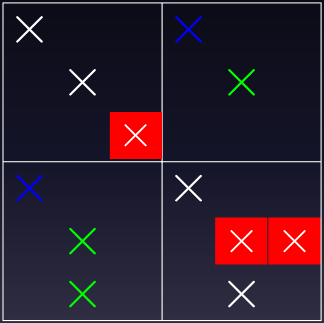
The P-fusion for the example



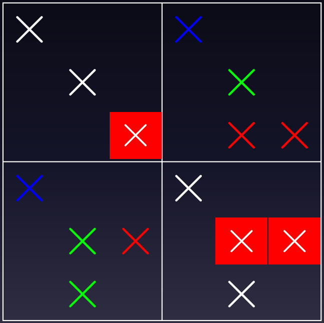
The P-fusion for the example



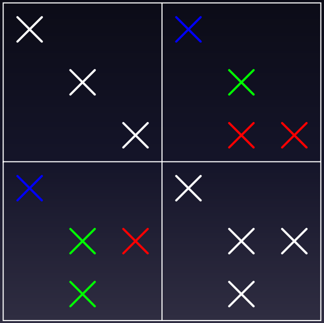
The P-fusion for the example



The P-fusion for the example



The P-fusion for the example



The P-fusion for the example

The P-fusion for the example

The P-fusion for the example