Essential Concepts

Thank you for coming to this experimental presentation.

The Executive Board of the CONCEPTS series asked me to deliver three short talks on “Essential Concepts”.

The idea is not to present new research results, but to illuminate some aspects of the theory which perhaps deserve more attention.

Today’s lecture is the most basic one. I’ll just look at a formal context and describe what I see.

Essential Concepts:

A cross is a cross

and a blank is a blank

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Formal Context

A **formal context** (G, M, I) consists of

a set G, the “objects”,

a set M, the “attributes”, and

a relation $I \subseteq G \times M$, the “incidence”.

Any finite formal context can be conveniently represented as a crosstable, in which

- the rows are labeled by the objects,
- the columns by the attributes,
- and a cross in row g and column m indicates that (g, m) belongs to the incidence relation I.

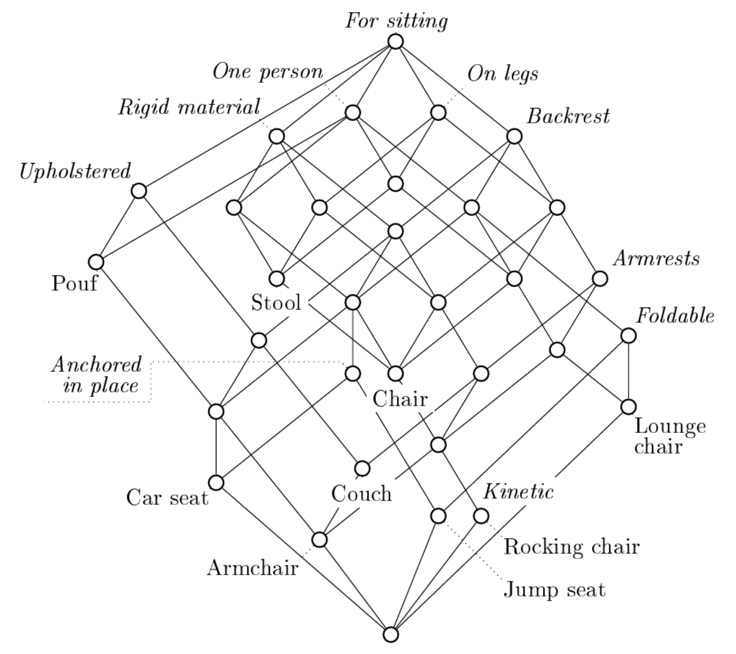
Note that one condition was not mentioned.

A lexical field

Seats	For sitting	On legs	For one person	Backrest	Armrests	Rigid material	Upholstered	Foldable	Anchored in place	Kinetic
Chair	×	×	×	×		×				
Armchair	×	×	×	×	×	×	×			
Stool	×	×	×			×				
Couch	×	×		×	×	×	×			
Pouf	×		×				×			
Jump seat	×		×	×		×		×	×	
Lounge chair	×	×	×	×	×			×		
Rocking chair	×	×	×	×	×	×				×
Car chair	×		×	×		×	×		×	

Based on : « *Vers une sémantique moderne* » by Bernard Pottier (1964), and on later extensions.

The concept lattice



Pottier's original data

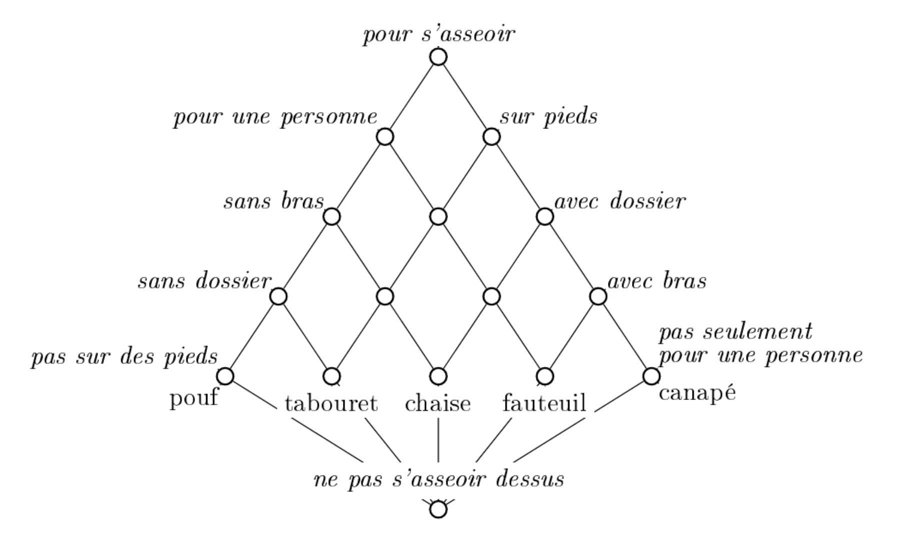
Seats (1964)	pour s'asseoir	sur pieds	pour une personne	avec dossier	avec bras
chaise	+	+	+	+	-
fauteuil	+	+	+	+	+
tabouret	+	+	+	-	-
canapé	+	+	-	+	+
pouf	+	-	+	-	-

This is a *many-valued context*. To obtain a concept lattice, *dichotomic scaling* is applied.

The derived context has ten attributes, two of which are reducible.

It has 16 formal concepts.

Pottier's lattice



Crosses and blanks

Seats	For sitting	On legs	For one person	Backrest	Armrests	Rigid material	Upholstered	Foldable	Anchored in place	Kinetic
Chair	×	×	×	×		×				
Armchair	×	×	×	×	×	×	×			
Stool	×	×	×			×				
Couch	×	×		×	×	×	×			
Pouf	×		×				×			
Jump seat	×		×	×		×		×	×	
Lounge chair	×	×	×	×	×			×		
Rocking chair	×	×	×	×	×	×				×
Car chair	×		×	×		×	×		×	

Except for the names there are only crosses and non-crosses (blanks).

Are all 50 crosses the same ?

Are all 40 blanks the same ?

Counting invariants

Seats	For sitting	On legs	For one person	Backrest	Armrests	Rigid material	Upholstered	Foldable	Anchored in place	Kinetic
Chair	×	×	×	×	5/4	×	5/4	5/2	5/2	5/1
Armchair	×	×	×	×	×	×	×	7/2	7/2	7/1
Stool	×	×	×	4/7	4/4	×	4/4	4/2	4/2	4/1
Couch	×	×	6/8	×	×	×	×	6/2	6/2	6/1
Pouf	×	3/6	×	3/7	3/4	3/7	×	3/2	3/2	3/1
Jump seat	×	6/6	×	×	6/4	×	6/4	×	×	6/1
Lounge chair	×	×	×	×	×	6/7	6/4	×	6/2	6/1
Rocking chair	×	×	×	×	×	×	7/4	7/2	7/2	×
Car chair	×	6/6	×	×	6/4	×	×	6/2	×	6/1

A context automorphism

PottierD	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
chaise	×		×		×		×			×
fauteuil	×		×		×		×		×	
tabouret	×		×		×			×		×
canapé	×		×			×	×		×	
pouf	×			×	×			×		×

PottierD	m_1	m_2	m_5	m_6	m_3	m_4	m_{10}	m_9	m_8	m_7
chaise	×		×		×		×			×
tabouret	×		×		×		×		×	
fauteuil	×		×		×			×		×
pouf	×		×			×	×		×	
canapé	×			×	×			×		×

The two formal contexts are equal, and so is the pattern of crosses and blanks. But the order of the objects was changed, and that of the attributes as well.
Automorphisms preserve invariants.

A theorem to prove

Conjecture : The following problem is graph isomorphism complete.

INSTANCE :	A formal context (G, M, I) and elements $(g, m), (h, n) \in I$.
QUESTION :	Is there a context automorphism mapping (g, m) to (h, n)?

Arrow relations

Seats	For sitting	On legs	For one person	Backrest	Armrests	Rigid material	Upholstered	Foldable	Anchored in place	Kinetic
Chair	×	×	×	×	⤢	×	↗	↗	↗	
Armchair	×	×	×	×	×	×	×	⤢	⤢	⤢
Stool	×	×	×	⤢		×	↗			
Couch	×	×	⤢	×	×	×	×			
Pouf	×	↗	×	⤢		⤢	×			
Jump seat	×	⤢	×	×	↙	×	⤢	×	×	↙
Lounge chair	×	×	×	×	×	⤢	⤢	×	↙	↙
Rocking chair	×	×	×	×	×	×	⤢	⤢	⤢	×
Car chair	×	⤢	×	×	↙	×	×	⤢	×	↙

Tight crosses

Definition :

(g, m) is a **tight incidence** of (G, M, I) iff

$$g \swarrow\!\!\!\nearrow m$$

holds for the **complementary context** $(G, M, G \times M \setminus I)$.

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	×	×	×	×		×				
	×	×	×	×	×	×	×			
	×	×	×			×				
	×	×		×	×	×	×			
	×		×				×			
	×		×	×		×		×	×	
	×	×	×	×	×			×		
	×	×	×	×	×	×				×
	×		×	×		×	×		×	

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	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×
	×	×	×	×	×	×	×	×	×	×

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					×		×	×	×	×
								×	×	×
				×	×		×	×	×	×
			×					×	×	×
		×		×	×	×		×	×	×
		×			×		×			×
						×	×		×	×
							×	×	×	
		×			×			×		×

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			$\swarrow\!\!\!\nearrow$	×		×	×	×	×
							×	×	×
	$\swarrow\!\!\!\nearrow$	$\swarrow\!\!\!\nearrow$	×	×	$\swarrow\!\!\!\nearrow$	×	×	×	×
		×		$\swarrow\!\!\!\nearrow$	$\swarrow\!\!\!\nearrow$	$\swarrow\!\!\!\nearrow$	×	×	×
	×	$\swarrow\!\!\!\nearrow$	×	×	×	$\swarrow\!\!\!\nearrow$	×	×	×
	×			×		×	$\swarrow\!\!\!\nearrow$	$\swarrow\!\!\!\nearrow$	×
				$\swarrow\!\!\!\nearrow$	×	×	$\swarrow\!\!\!\nearrow$	×	×
						×	×	×	$\swarrow\!\!\!\nearrow$
	×			×			×	$\swarrow\!\!\!\nearrow$	×

Tight crosses

Definition :

(g, m) is a **tight incidence** of (G, M, I) iff

$$g \swarrow\!\!\!\nearrow m$$

holds for the **complementary context** $(G, M, G \times M \setminus I)$.

			×	×		×	×	×	×
							×	×	×
	×	**×**	×	×	**×**	×	×	×	×
		×		**×**	**×**	**×**	×	×	×
	×	**×**	×	×	×	**×**	×	×	×
	×			×		×	**×**	**×**	×
				×	×	×	**×**	×	×
						×	×	×	**×**
	×			×			×	**×**	×

Tight crosses

Definition :

(g, m) is a **tight incidence** of (G, M, I) iff

$$g \mathrel{\swarrow\!\!\!\!\!\nearrow} m$$

holds for the **complementary context** $(G, M, G \times M \setminus I)$.

	×	×	×	**×**		×				
	×	×	×	×	×	×	×			
	×	**×**	**×**			**×**				
	×	×		×	**×**	**×**	**×**			
	×		**×**				**×**			
	×		×	×		×		**×**	**×**	
	×	×	×	×	**×**			**×**		
	×	×	×	×	×	×				**×**
	×		×	×		×	×		**×**	

. . . are useful

We say that a formal concept (A, B) **covers** an incidence $(g, m) \in I$ iff $(g, m) \in A \times B$.

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Theorem

A set $(A_1, B_1), (A_2, B_2), \ldots$ of formal concepts which covers all tight incidences covers all incidences.

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Theorem

A set $(A_1, B_1), (A_2, B_2), \ldots$ of formal concepts which covers all tight incidences covers all incidences.

This was used by Belohlavek, Outrata, and Trnecka in one of their algorithms for Boolean matrix factorization.

Instead of the greedily covering all crosses, their greedy algorithm covers all tight crosses ; the rest is then implied.

Tight incidences and arrows

Seats	For sitting	On legs	For one person	Backrest	Armrests	Rigid material	Upholstered	Foldable	Anchored in place	Kinetic
Chair	×	×	×	**×**	⤢	×	↗	↗	↗	
Armchair	×	×	×	×	×	×	×	⤢	⤢	⤢
Stool	×	**×**	**×**	⤢		**×**	↗			
Couch	×	×	⤢	×	**×**	**×**	**×**			
Pouf	×	↗	**×**	⤢		⤢	**×**			
Jump seat	×	⤢	×	×	↙	×	⤢	**×**	**×**	↙
Lounge chair	×	×	×	×	**×**	⤢	⤢	**×**	↙	↙
Rocking chair	×	×	×	×	×	×	⤢	⤢	⤢	**×**
Car chair	×	⤢	×	×	↙	×	×	⤢	**×**	↙

Why "tight"?

(g, m) is an incidence iff $\gamma g \leq \mu m$.

(g, m) is a tight incidence iff the interval $[\gamma g, \mu m]$ contains no further join- or meet-irreducible elements.